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Quantum metric fluctuations and Hawking radiation

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In this Rapid Communication we study the gravitational interactions between outgoing configurations giving rise to Hawking radiation and in-falling configurations. We exploit the fact that the fluctuations of the in-falling flux of energy across the horizon do not vanish in the vacuum state. This leads to interactions with outgoing quanta which grow near the horizon and prevent the appearance of trans-Planckian frequencies. These interactions express themselves in terms of metric fluctuations and lead to a description of Hawking radiation which is similar to that obtained from sound propagation in condensed matter models.

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In his original derivation [1], Hawking assumed that gravity can be treated classically; i.e. the metric was determined by the energy of the collapsing star but was unaffected by the quantum processes under examination. In this approximation, the radiation field satisfies a linear equation (in the absence of matter interactions) and the in-falling and outgoing field configurations are completely uncorrelated near the black hole horizon. In fact the pairs of quanta generated by its formation are composed of two outgoing quanta, one on each side of it. The external ones form the asymptotic flux, whereas their partners fall towards the singularity at $r=0$. Upon tracing over these inner configurations one gets an outgoing incoherent flux described by a thermal density matrix.

There is nevertheless a precise relationship between the expectation values of the in-falling and the outgoing energy fluxes. Indeed, the asymptotic thermal flux is accompanied by a negative in-falling flux $\langle T_{vv}\rangle$ which has, on the horizon, exactly the opposite value. Moreover $\langle T_{vv}\rangle$ drives black hole evaporation according to

$$\frac{dM}{dv}=\langle T_{vv}\rangle|_{r=r_{horizon}=2M}\simeq-\frac{1}{M^2}. \qquad (1)$$

We work with Planck units: $c=\hbar=M_{Planck}=1$. $v=t+r^*$ is the advanced null time and $r^*=r+2M\ln(r/2M-1)$ is the tortoise coordinate. $M(v)$ is the time dependent mass appearing in the Vaidya metric

$$ds^2=-\left(1-\frac{2M}{r}\right)dv^2+2dv\,dr+r^2(d\theta^2+\sin^2\theta d\phi^2) \qquad (2)$$

which describes the near horizon geometry ($|r-2M|\ll 2M$) of an evaporating black hole [2,3].

This semi-classical description would be perfectly valid if another feature of black hole physics wasn't present, namely, the field configurations giving rise to Hawking quanta possess arbitrary high (trans-Planckian) frequencies near the horizon: When measured by in-falling observers at r, the frequency grows as

$$\omega\propto\frac{\lambda}{1-2M/r} \qquad (3)$$

where λ is the asymptotic energy of the quantum. This implies that a wave packet centered along the null outgoing geodesic $u=t-r^*$ had a frequency $\omega\propto\lambda e^{\kappa u}$ when it emerged from the collapsing star. ($\kappa=1/4M$ is the surface gravity and fixes Hawking temperature $T_H=\kappa/2\pi$.) Unlike processes characterized by a typical energy scale, the relation $\omega\propto\lambda e^{\kappa u}$ shows that black hole evaporation rests on arbitrary high frequencies.

As emphasized by 't Hooft [4], this implies that the gravitational interactions between these configurations and in-falling quanta cannot be neglected, thereby questioning the validity of the semi-classical description. In questioning this validity, two issues should be distinguished (see Sec. 3.7 in [3]). First, there is the question of the low frequency $O(\kappa)$ changes which can be measured asymptotically, and second, that of the high frequency modifications of the near horizon physics. Since all thermo-dynamical reasonings indicate that the asymptotic properties (namely thermality governed by κ and stationarity) should be preserved, the problem is to conciliate their stability with the radical change of the near horizon physics which is needed to cure the trans-Planckian problem. Indeed, a perturbative analysis indicates that near horizon interactions lead to recoil effects proportional to ω, which seem incompatible with the stationarity of the flux [5].

A novel approach to this problem is provided by the analogy with condensed matter physics pointed out by Unruh [6]. He noticed that sound propagation in a moving fluid obeys a d'Alembertian equation which defines an acoustic metric. Therefore, when the acoustic metric corresponds to that of a collapsing star, thermally distributed phonons should be emitted. (The formation of an acoustic horizon occurs when the fluid velocity reaches the speed of sound.) However, contrary to photons the dispersion relation of phonons is not linear for frequencies (measured in the rest frame of the fluid) higher than a critical ω_c. Nevertheless, when $\omega_c\gg\kappa$, the asymptotic properties of Hawking phonons are unaffected [7] in spite of the fact that frequencies $\omega>\omega_c$ which were solicited in Hawking's derivation are no longer available.

This insensitivity suggests that something similar might apply to black holes. The aim of this Rapid Communication is to show that this is the case: The gravitational interactions between outgoing configurations giving rise to Hawking radiation and in-falling configurations in their ground state lead to collective effects which dissipate the trans-Planckian modes near the horizon, but without affecting the asymptotic properties of Hawking radiation. These collective effects express themselves in terms of a stochastic ensemble of metric fluctuations. The specification of the vacuum state at early times determines the statistical properties of this ensemble and this in turn fixes the cut-off ω_c (in terms of κ) and the frame which breaks the 2D Lorentz invariance [8,9].

For simplicity, we shall consider only s-waves propagating in spherically symmetric space times. The background metric results from the collapse of a null shell of mass M_0 which propagates along $v=0$ [10]. Inside the shell, for $v<0$, the geometry is Minkowski and described by Eq. (2) with $M=0$. Outside, the metric is also static and given by Eq. (2) with $M=M_0$.

When studying massless s-waves in this background, they fall into two classes according to their support on $\mathcal{J}^-$, the light-like past infinity. The waves in the first class have support only for $v<0$ and will be noted ϕ_-. They propagate inward in the flat geometry till $r=0$, where they bounce off and become outgoing configurations. This first class is itself divided in two: For $v<-4M_0$, the reflected waves cross the in-falling shell with $r>2M_0$ and reach the asymptotic region [11], whereas those for $0>v>-4M$ cross it with $r<2M_0$ and propagate in the trapped region till the singularity. The separating light ray $v_H=-4M_0$ becomes the future horizon $u=\infty$ after bouncing off at $r=0$. The configurations which form the second class have support only for $v>0$ and are noted ϕ_+. They are always in-falling and cross the horizon towards the singularity.

In the standard derivation of black hole radiation, the configurations for $v<v_H$ give rise to the asymptotic quanta, those for $v_H<v<0$ to their partners [12], whereas ϕ_+ plays no role in the asymptotic radiation. The correlations between the asymptotic quanta and their partners follow from the fact that, on $\mathcal{J}^-$ and in the vacuum, the rescaled field $\phi=r\chi$ (where χ is the 4D s-wave) satisfies

$$\langle\phi(v)\phi(v')\rangle=\int_0^\infty \frac{d\omega}{4\pi\omega}e^{-i\omega(v-v')}=-\frac{1}{4\pi}\ln|v-v'|. \tag{4}$$

Since this equation is valid for all v,v' there also exists correlations between ϕ_- and ϕ_+. However, they become negligible for late Hawking quanta since these emerge from configurations which are characterized by frequencies $\omega=\lambda e^{\kappa u}\gg\kappa$ and are localized very close to v_H. This follows from the asymptotic ($\kappa u\gg 1$) behavior of the relation between the value of u of the geodesic which originates from v on $\mathcal{J}^-$:

$$V(u)-v_H\propto e^{-\kappa u}. \tag{5}$$

As shown in [1], this exponential value induces both thermal radiation at temperature $\kappa/2\pi$ and the necessity of considering trans-Planckian frequencies on $\mathcal{J}^-$. In the absence of gravitational interactions, it also tells us that ϕ_- and ϕ_+ are effectively two independent fields.

Our aim now is to describe how the gravitational interactions between ϕ_- and ϕ_+ modify the semi-classical description of Hawking radiation. The generating functional of the whole system is

$$Z=\int \mathcal{D}\phi_+\,\mathcal{D}\phi_-\,\mathcal{D}h\; e^{i[S^{(+)}_{g+h}+S^{(-)}_{g+h}+S_{h,g}]}. \tag{6}$$

In this equation, h is the change of the metric with respect to the background g discussed above and $S_{h,g}$ is the action of h obtained from the Einstein-Hilbert action. $S^{(+)}_{g+h}$ and $S^{(-)}_{g+h}$ are the actions of ϕ_+ and ϕ_- propagating in the fluctuating geometry $g+h$.

When the metric fluctuations are spherically symmetric, h is characterized by two functions of v and r which are determined by the matter stress tensor. The line element in the fluctuating metric can be written as [10]

$$ds^2=(1+\psi)\left[-\left(1-\frac{2M}{r}\right)dv^2+2dv\,dr\right]+r^2d\Omega_2^2 \tag{7}$$

where $M=M_0+\mu(v,r)$. In this metric, the matter action is

$$S_{g+h}=-\int dv\,dr\left[\partial_v\phi\partial_r\phi+\left(1-\frac{2M}{r}\right)\frac{(\partial_r\phi)^2}{2}\right]. \tag{8}$$

It is independent of ψ, thereby showing a 2D conformal invariance (cf. [9,11]).

In the leading order in gravitational interactions, only quadratic terms in h should be kept in $S_{h,g}$. The Gaussian integration over h gives

$$Z=\int \mathcal{D}\phi_+\; e^{iS_g^{(+)}}\int \mathcal{D}\phi_-\; e^{iS_g^{(-)}+iS_{int}} \tag{9}$$

where S_{int} is a quadratic form of the stress tensor. It contains self-interaction terms depending on ϕ_- or ϕ_+ separately as well as a cross term. In the leading order, only the latter should be considered. Indeed, because of the 2D conformal invariance, the self-interaction terms vanish on-shell [13]. The cross term is given by [see Eq. (8)],

$$S_{int}=G\int_0^\infty dr\int_0^\infty dv\,\frac{\mu_+(v,r)}{r}(\partial_r\phi_-)^2 \tag{10}$$

where $\mu_+(v,r)$ is the mass fluctuation driven by ϕ_+ and G is Newton's constant. We have introduced it in the front of μ_+ to more easily read the order of the interactions between ϕ_- and ϕ_+. Since on-shell $(\partial_r\phi_-)^2\simeq(\partial_v\phi_-)^2/(r/2M-1)^2$, the dominant contribution to S_{int} is governed by μ_+ evaluated on the horizon:

$$\mu_+(v)=\mu_+(v,r)|_{r=2M}=\int_0^v dv'(\partial_{v'}\phi_+)^2. \tag{11}$$

In short, Eq. (10) represents the $\phi_-\phi_+$ interactions mediated by gravity. They have already been considered in [4,15,16]. The novelty of this Rapid Communication lies in their collective treatment when the state of ϕ_+ is the vacuum.

Before proceeding, let us first relate Eq. (9) to [1] and the semi-classical treatment. Hawking's approach is recovered by putting $G\mu_+ = 0$. Then ϕ_- is a free field propagating in the background geometry g, and ϕ_+ drops out from all matrix elements built with the operator ϕ_-. In this treatment one finds that matrix elements such as the in-out Green function are characterized by trans-Planckian frequencies when one of the operators approaches the horizon [12,10].

The semiclassical treatment is obtained from Eq. (9) by integrating over ϕ_+ and retaining only the mean $\langle \mu_+(v) \rangle$ driven by the (properly subtracted [3]) expectation of $T_{vv} = (\partial_v \phi_+)^2$,

$$\langle T_{vv}(v) \rangle|_{r=2M} = -\frac{\pi}{12}\left(\frac{\kappa}{2\pi}\right)^2 \tag{12}$$

evaluated in the unperturbed vacuum (4). This flux has the opposite value of a 2D thermal flux and drives black hole evaporation according to Eq. (1). Since one simply replaces M_0 by $M_0 + G\langle \mu_+ \rangle$ in the former treatment the matrix elements of ϕ_- are hardly affected [2] by the evaporation as long as it is slow, i.e. as long as $M(v) \gg M_{Planck}$. Therefore, the trans-Planckian problem stays as such in the semi-classical scenario.

To solve this problem clearly requires taking into account the *fluctuating* character of the interactions between ϕ_- and ϕ_+, which is encoded in the moments of T_{vv} higher than its mean (12). Upon considering matrix elements of ϕ_-, the integration over ϕ_+ in Eq. (9) will determine the influence functional (IF) [17] governing the effective dynamics of ϕ_-. To perform exactly this integration is out of reach since the final state of ϕ_+ will be correlated to that of ϕ_-. However, in the lowest order in G, this entanglement can be neglected. Indeed the back-reaction effects occurring through the modification of ϕ_+ are of higher order. This approximation is a common procedure both in quantum field theory where it gives the vacuum contribution (see Chap. 9 in [17]), and in statistical mechanics (e.g. the *polaron*, Chap. 11 in [17]). In our case the IF gives rise to a non-local action which is a sum of terms containing $(\partial_r \phi_-)^2$ and kernels given by the Wick contractions of T_{vv} evaluated with Eq. (4). The first term is quadratic in $(\partial_r \phi_-)^2$ and the kernel is the (connected) two-point function

$$\langle T_{vv}(v) T_{vv}(v') \rangle_c = \frac{1}{16\pi^2} \frac{1}{(v-v')^4}. \tag{13}$$

Using Eq. (11), one obtains

$$\langle \mu_+(v) \mu_+(v') \rangle = \frac{1}{96\pi^2} \frac{1}{(v-v')^2}$$
$$= \frac{1}{96\pi^2} \int_0^\infty d\omega \ \omega \cos[\omega(v-v')]. \tag{14}$$

This equation gives the mean metric fluctuations driven by ϕ_+ in the unperturbed $(G=0)$ vacuum state [18].

Keeping only this term in the IF is equivalent to work with a stochastic Gaussian ensemble of metric fluctuations [19]. Then, the conformal invariance of Eq. (8) allows us to obtain the non-linear effects [10] induced by these fluctuations from the characteristics of the equation for ϕ_-,

$$-\left(1 - \frac{2M_0 + 2G\mu_+(v)}{r}\right)\partial_r \phi_- = 2\partial_v \phi_-. \tag{15}$$

These are the outgoing null geodesics $u(v,r)$, solutions of $ds^2 = 0$ of Eq. (7). The background solution is $u_0 = v - 2r^*$. The first order change $\delta u = u - u_0$ is

$$\delta u(v)|_{u_0} = 2G \int_v^\infty dv' \frac{\mu_+(v')}{r(v')|_{u_0} - 2M_0} \tag{16}$$

where $r(v)|_{u_0}$ is obtained by inverting $u_0(v,r) = v - 2r^*$. The integral is dominated by the near horizon region where $r(v)|_{u_0} - 2M_0 \simeq 2M_0 e^{\kappa(v-u)}$. This dependence in κv will tame the UV content of the metric fluctuations.

To determine the physical effects of these fluctuations, let us analyze the asymptotic plane waves representing Hawking quanta since they govern the properties of the in-out Green function. In the absence of metric fluctuations the plane wave $e^{-i\lambda u}$ behaves as

$$e^{-i\lambda u_0(v,r)} = \theta(r - 2M_0) e^{-i\lambda v} (r - 2M_0)^{i\kappa\lambda}. \tag{17}$$

It vanishes for $r < 2M_0$ and possesses an infinite number of oscillations as $r \to 2M_0$ with increasing momentum $p_r = -i\partial_r$. This is the trans-Planckian problem.

When considering the Green function obtained from Eqs. (9), (10) and (13) (and with one operator at v,r and the other on $\mathcal{J}^+$), its behavior in v,r is determined [10] by the ensemble average waves

$$\langle e^{-i\lambda u(v,r)} \rangle \simeq e^{-i\lambda u_0(v,r)} e^{-\frac{\lambda^2}{2}\langle \delta u(v) \delta u(v) \rangle}. \tag{18}$$

Using Eqs. (14) and (16), one obtains

$$\langle \delta u(v)|_{u_0} \delta u(v)|_{u_0} \rangle = G^2 \int_0^\infty \frac{d\omega}{3} \frac{\kappa^2 \omega}{\kappa^2 + \omega^2} \frac{1}{(r/2M_0 - 1)^2}$$
$$= \sigma_\Lambda^2 \frac{1}{(r/2M_0 - 1)^2} \tag{19}$$

where the spread σ_Λ is equal to $G\kappa\sqrt{\ln(\Lambda/\kappa)/3}$. We have introduced the UV cut-off Λ to define the integral over ω. Notice that Λ is a Lorentz scalar (since it is the energy of an s-wave in its rest frame) and that its value is hardly relevant since $\sigma_{\Lambda=M} = \sqrt{2}\sigma_{\Lambda=1}$.

The main result of Eq. (19) is that σ_Λ is not proportional to Λ even though $\langle \mu_+^2 \rangle \simeq \Lambda^2$. This is because high frequencies $(\omega \gg \kappa)$ are damped by the integration over v' in Eq. (16). The frequencies $\omega \simeq \kappa$ dominate in Eq. (19).

Since $\langle \delta u \, \delta u \rangle$ diverges as $r \to 2M_0$, Eq. (18) tells us that the correlations between asymptotic quanta and early configurations, which existed in a given background as shown in Eq. (17), are washed out by the metric fluctuations once $r-2M_0 \simeq \sigma_\Lambda \simeq 1/M_0$. The physical reason for this loss of coherence is that the state of ϕ_+ becomes correlated to that of ϕ_- [4,15]. Phenomenologically this loss can be viewed as a dissipation of outgoing waves. Then, as in condensed matter [6,8], it can be included in the wave equation through a non-trivial dispersion relation.

Besides the dissipation of trans-Planckian correlations, we can verify that the gravitational interactions do not affect the asymptotic properties of Hawking radiation. This is achieved by computing the in-in Green function in the coincidence point limit. In this limit, one sees that the shift (16) drops out since it affects *both* points coherently. This cancellation guarantees that the asymptotic properties are unaffected [10].

In conclusion, we have studied the effects induced by the gravitational interactions governed by Eq. (10). Even though we worked out only the lowest order ($\sigma_\Lambda \propto G$) we believe that the following results are robust. (We see no reason for higher order terms to give $\sigma_\Lambda = 0$, thereby recovering trans-Planckian correlations. Indeed, when considering higher angular momentum modes, σ will be *larger* than our estimate [16].) (I) When propagated backwards in time, outgoing quanta are scattered by the metric fluctuations induced by in-falling matter fields in their vacuum state. (II) These interactions grow so strongly near the horizon that the quanta are completely scattered. (III) The in-falling vacuum fluctuations act as a reservoir of modes. This invites us to describe the interactions in terms of stochastic metric fluctuations. (IV) Even though the spectrum of the latter contains all frequencies (up to a UV cut-off), their impact on outgoing configurations is governed by frequencies $\omega \simeq \kappa$. (V) The stationarity of the vacuum [i.e. the fact that Eq. (13) is a function of $v-v'$ only] leads to stationary metric fluctuations (14) and this, combined with the stationarity of the background metric, gives a spread σ_Λ, which is independent of v.

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